

ON CONNECTED COMPONENT DECOMPOSITIONS OF QUANDLES

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ABSTRACT. We give a formula of the connected component decomposition of the Alexander quandle: $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \dots, f_k(t)) = \bigsqcup_{i=0}^{a-1} \text{Orb}(i)$, where $a = \gcd(f_1(1), \dots, f_k(1))$. We show that the connected component $\text{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/J$ with an explicit ideal J . By using this, we see how a quandle is decomposed into connected components for some Alexander quandles. We introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles. In some cases, this decomposition is obtained by iterating a connected component decomposition. We also discuss the maximal connected sub-multiple conjugation quandle decomposition.

1. INTRODUCTION

A quandle [5, 6] is an algebraic structure whose axioms are derived from the Reidemeister moves on oriented link diagrams. An inner automorphism group of a quandle has an action to the quandle naturally. We call an orbit of the quandle by the action its connected component, which is a subquandle. A quandle is said to be connected if the action is transitive. It is known that all connected quandles of prime square order are Alexander quandles [2].

Any connected component of an Alexander quandle M is isomorphic to $(1-t)M$. S. Nelson [7] proved that two finite Alexander quandles M and N of the same cardinality are isomorphic if and only if $(1-t)M$ and $(1-t)N$ are isomorphic as modules, and showed connectivity of some Alexander quandles. The numbers of Alexander quandles and connected ones are listed up to order 16 in [7, 8]. In this paper, for any $f_1(t), \dots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$, we show that the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \dots, f_k(t))$ is $\bigsqcup_{i=0}^{a-1} \text{Orb}(i)$ and that $\text{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/J$ with an explicit ideal J , where $a = \gcd(f_1(1), \dots, f_k(1))$.

A connected subquandle has played an important roll in colorings of a knot diagram. However a connected component of a quandle is not a connected quandle in general. In this paper, we introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles, and show that it is obtained by iterating a connected component decomposition when the quandle is finite, where we note that the decomposition of a finite quandle obtained by iterating a connected component decomposition was introduced in [1, 9]. We also give examples of the decompositions of some quandles. For example, we concretely determine the decompositions of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ for any $n_0 \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$ and the dihedral quandle R_m for any $m \in \mathbb{Z}_{\geq 0}$.

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We also discuss the similar decomposition of a multiple conjugation quandle, which is an algebraic structure whose axioms are derived from the Reidemeister moves on diagrams of spatial trivalent graphs and handlebody-links.

This paper is organized as follows. In Section 2, we recall the definition of a quandle and its connected components. In Section 3, we determine the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), \dots, f_k(t))$. In Section 4, we introduce a decomposition of a quandle into the disjoint union of maximal connected subquandles and show that it is obtained by iterating a connected component decomposition when the quandle is finite. In Section 5, we give examples of the maximal connected subquandle decompositions of some quandles. In Section 6, we recall the definition of a multiple conjugation quandle and introduce its properties we use in Section 7. In Section 7, we discuss the similar decomposition of a multiple conjugation quandle.

2. A QUANDLE

A *quandle* X is a non-empty set with a binary operation $*$: $X \times X \rightarrow X$ satisfying the following axioms.

- For any $a \in X$, $a * a = a$.
- For any $a \in X$, the map $S_a : X \rightarrow X$ defined by $S_a(x) = x * a$ is a bijection.
- For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

We give some examples of quandles. Let G be a group. We define a binary operation $*$: $G \times G \rightarrow G$ by $a * b = b^{-1}ab$ for any $a, b \in G$. Then G is a quandle, which is called a *conjugation quandle* and denoted by $\text{Conj}(G)$. The second example is obtained from a $\mathbb{Z}[t^{\pm 1}]$ -module M . We define a binary operation $*$: $M \times M \rightarrow M$ by $a * b = ta + (1 - t)b$ for any $a, b \in M$. Then M is a quandle, which is called an *Alexander quandle*. The third example is a *dihedral quandle* $R_m := \mathbb{Z}/m\mathbb{Z}$ for any $m \in \mathbb{Z}_{\geq 0}$. We define a binary operation $*$: $R_m \times R_m \rightarrow R_m$ by $a * b = 2b - a$ for any $a, b \in R_m$. Then R_m is a quandle.

Let $(X, *)$ be a quandle. In this paper, we write $a_1 * a_2 * \dots * a_n$ for $(\dots((a_1 * a_2) * a_3) * \dots * a_n)$ simply. For any $a \in X$ and $n \in \mathbb{Z}_{\geq 0}$, $S_a^n : X \rightarrow X$ and $S_a^{-n} : X \rightarrow X$ are n -th functional powers of S_a and S_a^{-1} respectively. For any $a, b \in X$ and $n \in \mathbb{Z}$, we define a binary operation $*^n : X \times X \rightarrow X$ by $a *^n b = S_b^n(a)$. Then $(X, *^n)$ is also a quandle. We define the *type* of X by the minimal number of n satisfying $a *^n b = a$ for any $a, b \in X$.

Let $(X, *_X)$ and $(Y, *_Y)$ be quandles. A *homomorphism* $\phi : X \rightarrow Y$ is a map from X to Y satisfying $\phi(x *_X y) = \phi(x) *_Y \phi(y)$ for any $x, y \in X$. We call a bijective homomorphism an *isomorphism*. X and Y are *isomorphic*, denoted $X \cong Y$, if there exists an isomorphism from X to Y . We call an isomorphism from X to X an *automorphism* of X . For any $a \in X$ and $n \in \mathbb{Z}$, the map $S_a^n : X \rightarrow X$ is an automorphism of X .

Let $(X, *)$ be a quandle. A non-empty subset Y of X is called a *subquandle* of X if Y itself is a quandle under $*$. For any subset Y of X , Y is a subquandle of X if and only if $a * b, a *^{-1} b \in Y$ for any $a, b \in Y$.

For any subset A of X , the minimal subquandle of X including A , denoted by $\langle A \rangle$, is called the subquandle generated by A , that is,

$$\langle A \rangle = \{a *^{k_1} x_1 *^{k_2} \dots *^{k_n} x_n \in X \mid a, x_1, \dots, x_n \in A, k_1, k_2, \dots, k_n \in \mathbb{Z}\}.$$

Let X be a quandle. All automorphisms of X form a group under composition of morphisms: $f \cdot g := g \circ f$. This group is called the *automorphism group* of X and denoted by $\text{Aut}(X)$. For a subset A of X , we denote by $\text{Inn}(A)$ a subgroup of $\text{Aut}(X)$ generated by $\{S_a \mid a \in A\}$. In particular, $\text{Inn}(X)$ is called the *inner automorphism group* of X . For any $a \in X$ and $g \in \text{Inn}(A)$, we define an action of $\text{Inn}(A)$ on X by $a \cdot g = g(a)$. We say that X is a *connected quandle* when the action is transitive. In general, an orbit of X by the action is called a *connected component* of X or an orbit of X simply, and $X = \bigsqcup_{i \in I} X_i$ is called the *connected component decomposition* of X when X_i is a connected component of X for any $i \in I$. In general, a connected component of X is a subquandle of X . We denote by $\text{Orb}_X(a)$ or $\text{Orb}(a)$ the orbit of X containing a .

Example 1. For any group G , a connected component of $\text{Conj}(G)$ coincides with one of conjugacy classes of G .

In the following, we give a well-known fact with a proof.

Lemma 1. Let M be an Alexander quandle. Then any connected component of M is isomorphic to $(1-t)M$.

Proof. Since $0 * x = (1-t)x$ for any $x \in M$, it follows that $(1-t)M \subset \text{Orb}(0)$. On the other hand, for any $z \in \text{Orb}(0)$, there exist $y_1, \dots, y_n \in M$ such that $z = 0 * y_1 * y_2 * \dots * y_n = (1-t)y_1 * y_2 * \dots * y_n$. Since for any $x, y \in M$, $(1-t)x * y = t(1-t)x + (1-t)y = (1-t)(tx + y)$, we have $z \in (1-t)M$, that is, $(1-t)M \supset \text{Orb}(0)$. Therefore $(1-t)M = \text{Orb}(0)$. Next, we define the map $\phi_a : \text{Orb}(0) \rightarrow \text{Orb}(a)$ by $\phi_a(x) = x + a$ for any $a \in M$. For any $(1-t)x \in \text{Orb}(0)$ and $a \in M$, $\phi_a((1-t)x) = (1-t)x + a = ta + (1-t)(x + a) = a * (x + a) \in \text{Orb}(a)$. Hence ϕ_a is well-defined. For any $x, y \in \text{Orb}(0)$ and $a \in M$, $\phi_a(x * y) = (tx + (1-t)y) + a = t(x + a) + (1-t)(y + a) = \phi_a(x) * \phi_a(y)$. Hence ϕ_a is a homomorphism. Similarly, the map $\psi_a : \text{Orb}(a) \rightarrow \text{Orb}(0)$ defined by $\psi_a(x) = x - a$ is a homomorphism for any $a \in M$. Since $\phi_a \circ \psi_a = \text{id}_{\text{Orb}(a)}$ and $\psi_a \circ \phi_a = \text{id}_{\text{Orb}(0)}$, ϕ_a is an isomorphism. Therefore $\text{Orb}(0)$ and $\text{Orb}(a)$ are isomorphic for any $a \in M$. \square

3. THE CONNECTED COMPONENT DECOMPOSITION OF AN ALEXANDER QUANDLE

In this section, we show that the connected component decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ is $\bigsqcup_{i=0}^{a-1} \text{Orb}(i)$, where $(f_1(t), f_2(t), \dots, f_k(t))$ is an ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by Laurent polynomials $f_1(t), f_2(t), \dots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$, and $a = \gcd(f_1(1), f_2(1), \dots, f_k(1))$. Furthermore, we determine the form of all connected components of $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$.

For any $\alpha(t) \in \mathbb{Z}[t^{\pm 1}]$, we define $C_{\alpha(t)}$ by

$$C_{\alpha(t)} = \{\alpha(t) + a_1 f_1(t) + a_2 f_2(t) + \dots + a_k f_k(t) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}[t^{\pm 1}]\} \subset \mathbb{Z}[t^{\pm 1}].$$

Then for any $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, $C_{[\alpha(t)]} := C_{\alpha(t)}$ is well-defined. In this paper, we often write $\alpha(t)$ for $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ simply. For any $D \subset \mathbb{Z}[t^{\pm 1}]$, we define $D(1)$ by $D(1) = \{g(1) \mid g(t) \in D\} \subset \mathbb{Z}$. It is easy to see that

$$C_{[\alpha(t)]}(1) = \{\alpha(1) + a_1 f_1(1) + a_2 f_2(1) + \dots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\}$$

for any $[\alpha(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$.

Lemma 2. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, it follows that $C_{[\alpha(t)]*[\beta(t)]}(1) = C_{[\alpha(t)]}(1)$.*

Proof. Since $[\alpha(t)] * [\beta(t)] = [t\alpha(t) + (1-t)\beta(t)]$, we have

$$\begin{aligned} C_{[\alpha(t)]*[\beta(t)]}(1) &= \{1 \cdot \alpha(1) + (1-1)\beta(1) + a_1 f_1(1) + \dots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\} \\ (2) \quad &= \{\alpha(1) + 0 \cdot \beta(1) + a_1 f_1(1) + \dots + a_k f_k(1) \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\} \\ &= C_{[\alpha(t)]}(1). \end{aligned}$$

□

Lemma 3. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, it follows that $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$ if and only if $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$.*

Proof. Suppose that $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. Then we have

$$\begin{aligned} (3) \quad &\{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\ (4) \quad &= C_{[\alpha(t)]}(1) \\ (5) \quad &= C_{[\beta(t)]}(1) \\ &= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\}. \end{aligned}$$

Hence there exist $l_1, l_2, \dots, l_k \in \mathbb{Z}$ such that $\alpha(1) = \beta(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1)$. We put $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ by $\alpha(t) = (1-t)\tilde{\alpha}(t) + \alpha(1)$ and $\beta(t) = (1-t)\tilde{\beta}(t) + \beta(1)$ respectively. We also put $\tilde{f}_i(t)$ by $f_i(t) = (1-t)\tilde{f}_i(t) + f_i(1)$ for any $i = 1, 2, \dots, k$ and $\gamma(t) := \tilde{\beta}(t) + \beta(1) - t\tilde{\alpha}(t) + \sum_{i=1}^k l_i t \tilde{f}_i(t)$. Since $[\sum_{i=1}^k l_i t f_i(t)] = [0]$ in $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, we have

$$\begin{aligned} (6) \quad &[(1-t)\gamma(t)] \\ (7) \quad &= [(1-t)(\tilde{\beta}(t) + \beta(1) - t\tilde{\alpha}(t) + \sum_{i=1}^k l_i t \tilde{f}_i(t))] \\ (8) \quad &= [(1-t)\tilde{\beta}(t) + (1-t)\beta(1) - t(1-t)\tilde{\alpha}(t) + \sum_{i=1}^k l_i t(1-t)\tilde{f}_i(t)] \\ (9) \quad &= [(1-t)\tilde{\beta}(t) + (1-t)\beta(1) - t(1-t)\tilde{\alpha}(t) + \sum_{i=1}^k l_i t(1-t)\tilde{f}_i(t) - \sum_{i=1}^k l_i t f_i(t)] \\ (10) \quad &= [(1-t)\tilde{\beta}(t) + \beta(1) - t\beta(1) - t(1-t)\tilde{\alpha}(t) + \sum_{i=1}^k l_i t((1-t)\tilde{f}_i(t) - f_i(t))] \\ (11) \quad &= [\beta(t) - t(1-t)\tilde{\alpha}(t) - t\beta(1) + \sum_{i=1}^k l_i t(-f_i(1))] \\ (12) \quad &= [\beta(t) - t(1-t)\tilde{\alpha}(t) - t\alpha(1)] \\ &= [\beta(t) - t\alpha(t)]. \end{aligned}$$

Hence $[\alpha(t)] * [\gamma(t)] = [t\alpha(t) + (1-t)\gamma(t)] = [\beta(t)]$, which implies $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$. Next, suppose that $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$. There exist $[\gamma_1(t)], [\gamma_2(t)], \dots, [\gamma_l(t)] \in \mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ and $\epsilon_1, \epsilon_2, \dots, \epsilon_l \in \mathbb{Z}$ such that $[\alpha(t)] *^{\epsilon_1} [\gamma_1(t)] *^{\epsilon_2} \dots *^{\epsilon_l} [\gamma_l(t)] = [\beta(t)]$. By Lemma 2, we have $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. □

Then we have the following lemma.

Lemma 4. *For any elements $[\alpha(t)]$ and $[\beta(t)]$ of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$, it follows that $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$ if and only if $\alpha(1) \equiv \beta(1) \pmod{a}$, where $a = \gcd(f_1(1), f_2(1), \dots, f_k(1))$.*

Proof. Suppose that $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$. By Lemma 3, $C_{[\alpha(t)]}(1) = C_{[\beta(t)]}(1)$. Since $f_1(1)\mathbb{Z} + f_2(1)\mathbb{Z} + \dots + f_k(1)\mathbb{Z} = a\mathbb{Z}$, we have

$$\begin{aligned} (13) \quad C_{[\alpha(t)]}(1) + la \mid l \in \mathbb{Z} &= \{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\ (14) &= C_{[\alpha(t)]}(1) \\ (15) &= C_{[\beta(t)]}(1) \\ (16) &= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\ &= \{\beta(1) + la \mid l \in \mathbb{Z}\}. \end{aligned}$$

Hence we obtain $\alpha(1) \equiv \beta(1) \pmod{a}$. On the other hand, suppose that $\alpha(1) \equiv \beta(1) \pmod{a}$. Since $f_1(1)\mathbb{Z} + f_2(1)\mathbb{Z} + \dots + f_k(1)\mathbb{Z} = a\mathbb{Z}$, we have

$$\begin{aligned} (17) \quad C_{[\alpha(t)]}(1) &= \{\alpha(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\ (18) &= \{\alpha(1) + la \mid l \in \mathbb{Z}\} \\ (19) &= \{\beta(1) + la \mid l \in \mathbb{Z}\} \\ (20) &= \{\beta(1) + l_1 f_1(1) + l_2 f_2(1) + \dots + l_k f_k(1) \mid l_1, l_2, \dots, l_k \in \mathbb{Z}\} \\ &= C_{[\beta(t)]}(1). \end{aligned}$$

By Lemma 3, we obtain $\text{Orb}([\alpha(t)]) = \text{Orb}([\beta(t)])$. □

Then we have the following theorem.

Theorem 1. *Let M be the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ and let $a = \gcd(f_1(1), f_2(1), \dots, f_k(1))$. Then the following hold.*

(1) *The connected component decomposition of M is given by*

$$M = \bigsqcup_{i=0}^{a-1} \text{Orb}(i),$$

where

$$\begin{aligned} \text{Orb}(i) &= \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], g(1) \equiv i \pmod{a}\} \\ &= \{[i + (1-t)g(t) + aj] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], j \in \mathbb{Z}\}. \end{aligned}$$

(2) *For any $j = 0, 1, \dots, a-1$, it follows that*

$$\text{Orb}(j) \cong \mathbb{Z}[t^{\pm 1}]/((f_1(t), f_2(t), \dots, f_k(t)) + I),$$

where we define $\tilde{f}_i(t)$ by $f_i(t) = (1-t)\tilde{f}_i(t) + f_i(1)$ for any $i = 1, 2, \dots, k$, and $I = \{\sum_{i=1}^k a_i \tilde{f}_i(t) \mid a_i \in \mathbb{Z}[t^{\pm 1}], \sum_{i=1}^k a_i f_i(1) = 0\}$.

Proof. (1) By Lemma 4, $M = \bigsqcup_{i=0}^{a-1} \text{Orb}(i)$ is the connected component decomposition of M , and we have $\text{Orb}(i) = \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], g(1) \equiv i \pmod{a}\}$ immediately. There exist $\tilde{g}(t) \in \mathbb{Z}[t^{\pm 1}]$ and $j \in \mathbb{Z}$ such that $g(t) = g(1) + (1-t)\tilde{g}(t) = i + (1-t)\tilde{g}(t) + aj$ if and only if $g(1) \equiv i \pmod{a}$. Hence we have $\text{Orb}(i) = \{[i + (1-t)g(t) + aj] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], j \in \mathbb{Z}\}$.

- (2) Let $J = (f_1(t), f_2(t), \dots, f_k(t))$. We define the $\mathbb{Z}[t^{\pm 1}]$ -homomorphism $\phi : \mathbb{Z}[t^{\pm 1}] \rightarrow (1-t)M$ by $\phi(x) = (1-t)x + J$. It is clear that $J \subset \ker(\phi)$. For any $\sum_{i=1}^k a_i \tilde{f}_i(t) \in I$, we have

$$\begin{aligned}
 (21) \quad \phi\left(\sum_{i=1}^k a_i \tilde{f}_i(t)\right) &= (1-t) \sum_{i=1}^k a_i \tilde{f}_i(t) + J \\
 (22) \quad &= \sum_{i=1}^k a_i (1-t) \tilde{f}_i(t) + \sum_{i=1}^k a_i f_i(1) + J \\
 (23) \quad &= \sum_{i=1}^k a_i ((1-t) \tilde{f}_i(t) + f_i(1)) + J \\
 &= \sum_{i=1}^k a_i f_i(t) + J,
 \end{aligned}$$

which implies that $I \subset \ker(\phi)$. Hence we obtain $J + I \subset \ker(\phi)$. On the other hand, let $g(t) \in \ker(\phi)$. Since $\phi(g(t)) = (1-t)g(t) + J = J$, there exist $h_1(t), h_2(t), \dots, h_k(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $(1-t)g(t) = \sum_{i=1}^k h_i(t) f_i(t)$. When we put $t = 1$, we have $0 = \sum_{i=1}^k h_i(1) f_i(1)$. For any $i = 1, 2, \dots, k$, we define $\tilde{h}_i(t)$ by $h_i(t) = (1-t)\tilde{h}_i(t) + h_i(1)$. Since $\sum_{i=1}^k h_i(1) f_i(1) = 0$, we have

$$\begin{aligned}
 (24) \quad (1-t)g(t) &= \sum_{i=1}^k h_i(t) f_i(t) \\
 (25) \quad &= \sum_{i=1}^k ((1-t)\tilde{h}_i(t) f_i(t) + (1-t)h_i(1)\tilde{f}_i(t) + h_i(1)f_i(1)) \\
 (26) \quad &= (1-t) \sum_{i=1}^k \tilde{h}_i(t) f_i(t) + (1-t) \sum_{i=1}^k h_i(1)\tilde{f}_i(t) + \sum_{i=1}^k h_i(1)f_i(1) \\
 &= (1-t) \sum_{i=1}^k \tilde{h}_i(t) f_i(t) + (1-t) \sum_{i=1}^k h_i(1)\tilde{f}_i(t).
 \end{aligned}$$

Hence we have $g(t) = \sum_{i=1}^k \tilde{h}_i(t) f_i(t) + \sum_{i=1}^k h_i(1)\tilde{f}_i(t)$. Since $\sum_{i=1}^k h_i(1)f_i(1) = 0$, we have $\sum_{i=1}^k h_i(1)\tilde{f}_i(t) \in I$, that is, $g(t) \in J + I$. Hence we obtain $J + I \supset \ker(\phi)$. Obviously, ϕ is a surjection. By the homomorphism theorem, $\tilde{\phi} : \mathbb{Z}[t^{\pm 1}]/(J + I) \rightarrow (1-t)M$ is a $\mathbb{Z}[t^{\pm 1}]$ -isomorphism, which is an isomorphism as quandles. By Lemma 1, it follows that $\text{Orb}(j) \cong \mathbb{Z}[t^{\pm 1}]/((f_1(t), f_2(t), \dots, f_k(t)) + I)$ for any $j = 0, 1, \dots, a-1$. \square

By Theorem 1, we obtain the following corollaries, where we note that the dihedral quandle R_m is isomorphic to the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(m, t+1)$ for any $m \in \mathbb{Z}_{\geq 0}$.

Corollary 1. *For any $m \in \mathbb{Z}_{>0}$, R_m is a connected dihedral quandle if and only if m is an odd number. Furthermore, when m is an even number, $R_m = \text{Orb}(0) \sqcup \text{Orb}(1) = \{0, 2, \dots, m-2\} \sqcup \{1, 3, \dots, m-1\}$ is the connected component decomposition of R_m .*

Corollary 2. *Let $f_1(t), f_2(t), \dots, f_k(t) \in \mathbb{Z}[t^{\pm 1}]$. Then the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(f_1(t), f_2(t), \dots, f_k(t))$ is connected if and only if $\gcd(f_1(1), f_2(1), \dots, f_k(1)) = 1$.*

For example, the tetrahedral quandle $\mathbb{Z}[t^{\pm 1}]/(2, t^2 + t + 1)$ is connected by Corollary 2.

Corollary 3. *Let $a, m \in \mathbb{Z}$ and let $n = m/\gcd(m, 1+a)$. Then any connected component of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(m, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n, t+a)$.*

Proof. By Theorem 1, any connected component of $\mathbb{Z}[t^{\pm 1}]/(m, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((m, t+a) + I)$, where $I = \{-a_2 \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}], a_1 m + a_2(1+a) = 0\}$. For any $-x_2 \in I$, there exists $x_1 \in \mathbb{Z}[t^{\pm 1}]$ such that $x_1 m + x_2(1+a) = 0$. Then we have $x_1 n + x_2(1+a)/\gcd(m, 1+a) = 0$. Since n and $(1+a)/\gcd(m, 1+a)$ are relatively prime, x_2 is divisible by n . Hence we obtain $I \subset n\mathbb{Z}[t^{\pm 1}]$. On the other hand, for any $x_2 = ns \in n\mathbb{Z}[t^{\pm 1}]$, there exists $x_1 = -s(1+a)/\gcd(m, 1+a) \in \mathbb{Z}[t^{\pm 1}]$ such that $x_1 m + x_2(1+a) = 0$. Hence $-x_2 \in I$, that is, $I \supset n\mathbb{Z}[t^{\pm 1}]$. Therefore $I = n\mathbb{Z}[t^{\pm 1}]$. Since m is divisible by n , we have $\mathbb{Z}[t^{\pm 1}]/((m, t+a) + I) = \mathbb{Z}[t^{\pm 1}]/(m, n, t+a) = \mathbb{Z}[t^{\pm 1}]/(n, t+a)$. \square

Corollary 4. *If m is an even number, then any connected component of R_m is isomorphic to $R_{m/2}$.*

4. THE MAXIMAL CONNECTED SUBQUANDLE DECOMPOSITION

In this section, we consider a decomposition of a quandle into the disjoint union of maximal connected subquandles, and show that it is uniquely obtained by iterating a connected component decomposition when the quandle is finite. We remark that $\{(1, 2, 3), (1, 3, 2)\}$ is a connected component of $\text{Conj}(S_3)$, but not a connected subquandle of it, where S_3 is a symmetric group of degree 3.

Let X be a quandle and let A be a connected subquandle of X . We say that A is a *maximal connected subquandle* of X when any connected subquandle of X including A is only A . We say that $X = \bigsqcup_{i \in I} A_i$ is the *maximal connected subquandle decomposition* of X when each A_i is a maximal connected subquandle of X .

Let X be a quandle and let A be a subset of X . For any $a, b \in X$, we write $a \sim_A b$ when there exists $f \in \text{Inn}(A)$ such that $f(a) = b$. It is an equivalence relation on X . It is easy to see that $a \sim_A b$ if and only if there exist $a_1, a_2, \dots, a_n \in A$ and $k_1, k_2, \dots, k_n \in \mathbb{Z}$ such that $a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n = b$. Furthermore X is a connected quandle if and only if $a \sim_X b$ for any $a, b \in X$.

Lemma 5. *Let X be a quandle and let A_i be connected subquandles of X for any $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\langle \bigcup_{i \in I} A_i \rangle$ is a connected subquandle of X .*

Proof. Let $A = \bigcup_{i \in I} A_i$, and suppose that $\bigcap_{i \in I} A_i \neq \emptyset$. For any $x, y \in \langle A \rangle$, there exist $a, b \in A$ such that $x \sim_A a$ and $y \sim_A b$. For any $c \in \bigcap_{i \in I} A_i$, we have $a \sim_A c$ and $b \sim_A c$. Hence we have $a \sim_A b$, which implies that $x \sim_A y$. Therefore $x \sim_{\langle A \rangle} y$, that is, $\langle A \rangle$ is a connected subquandle of X . \square

Lemma 6. *Let X be a quandle and let A_1 and A_2 be maximal connected subquandles of X . If $A_1 \cap A_2 \neq \emptyset$, then $A_1 = A_2$.*

Proof. Suppose that $A_1 \cap A_2 \neq \emptyset$. By Lemma 5, $\langle A_1 \cup A_2 \rangle$ is a connected subquandle of X . Since A_1 and A_2 are included in $\langle A_1 \cup A_2 \rangle$ and maximal connected subquandles of X , we obtain $\langle A_1 \cup A_2 \rangle = A_1 = A_2$. \square

Theorem 2. *Any quandle has the unique maximal connected subquandle decomposition.*

Proof. Let X be a quandle. For any $a \in X$, we define $[a] := \bigcup_{a \in W} W$, where W is a connected subquandle of X . By Lemma 5, $\langle [a] \rangle$ is a connected subquandle of X . Suppose that A is a connected subquandle of X including $\langle [a] \rangle$. Since A contains a , we have $A \subset [a]$, that is, $A = [a] = \langle [a] \rangle$. Hence $[a]$ is a maximal connected subquandle of X . For any $a, b \in X$, we have $[a] \cap [b] = \emptyset$ or $[a] = [b]$ by Lemma 6. Therefore there exists a subset Y of X such that $X = \bigsqcup_{a \in Y} [a]$. It is the maximal connected subquandle decomposition of X . Next, we show the uniqueness. Let B be a maximal connected subquandle of $X = \bigsqcup_{a \in Y} [a]$. Then there exists $a' \in Y$ such that $B \cap [a'] \neq \emptyset$. By Lemma 6, we have $B = [a']$. Therefore X has the unique maximal connected subquandle decomposition. \square

For a quandle X , it is easy to see that any connected subquandle of X is included in some connected component of X . Therefore if a connected component of X is a connected subquandle of X , then it is a maximal connected subquandle of X .

Let X be a quandle and let $\mathcal{P}_{\text{Qnd}}(X)$ be the set of all subquandles of X . For any $\mathcal{A} \subset \mathcal{P}_{\text{Qnd}}(X)$, we define $D(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \{\text{Orb}_A(a) \mid a \in A\}$. It is easy to see that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in D(\mathcal{A})} A$. We put $D^0(\mathcal{A}) := \mathcal{A}$ and $D^{k+1}(\mathcal{A}) := D(D^k(\mathcal{A}))$ for any $k \in \mathbb{Z}_{\geq 0}$.

Theorem 3. *Let X be a quandle. If there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(\{X\}) = D^{n+1}(\{X\})$, then $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X . In particular, if X is a finite quandle, then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X .*

Proof. Suppose that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(\{X\}) = D^{n+1}(\{X\})$. For any $A \in D^n(\{X\})$, A has only one orbit A , that is, A is a connected subquandle of X . Let Y be a connected subquandle of X including A and let $X_0 = X$. Then there exists an orbit of X_0 including Y . We denote by $X_1 \in D^1(\{X\})$ the orbit. For any $i \in \mathbb{Z}_{\geq 0}$, we denote by $X_{i+1} \in D^{i+1}(\{X\})$ an orbit of X_i including Y inductively. Then we have $A \subset Y \subset X_n$. Since $A, X_n \in D^n(\{X\})$, we have $A = Y = X_n$. Therefore A is the maximal connected subquandle of X , and $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X . Next, let X be a finite quandle. For any $i \in \mathbb{Z}_{\geq 0}$, we have $\#D^i(\{X\}) \leq \#D^{i+1}(\{X\}) \leq \#X$. Since $\#X$ is finite, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\#D^n(\{X\}) = \#D^{n+1}(\{X\})$, which implies that $D^n(\{X\}) = D^{n+1}(\{X\})$. Therefore $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X . \square

By Lemma 1 and Theorem 3, the following corollary holds immediately.

Corollary 5. *All maximal connected subquandles of a finite Alexander quandle are isomorphic.*

For a quandle X , we denote by $\text{depth}(X)$ the minimal number of n satisfying $X = \bigsqcup_{A \in D^n(\{X\})} A$ is the maximal connected subquandle decomposition of X . It

is called the *subquandle depth* of X in [9]. Obviously, X is a connected quandle if and only if $\text{depth}(X) = 0$.

5. EXAMPLES OF THE MAXIMAL CONNECTED SUBQUANDLE DECOMPOSITION

In this section, we give examples of the maximal connected subquandle decompositions of some quandles.

Let S_n be a symmetric group of degree n . We consider connectivity of $\text{Conj}(S_n)$. By Example 1, a connected component of $\text{Conj}(S_n)$ coincides with one of conjugacy classes of S_n . We denote by $C(a)$ the conjugacy class of S_n containing a . We note that two elements of S_n are conjugate if and only if their cyclic types coincide.

Example 2. (1) *We show that the maximal connected subquandle decomposition of S_3 is*

$$S_3 = \{(123)\} \sqcup \{(132)\} \sqcup C((12)) \sqcup C(e).$$

$S_3 = C((123)) \sqcup C((12)) \sqcup C(e)$ is the connected component decomposition of S_3 . Furthermore, $C((12))$ and $C(e)$ are connected quandles, and $C((123)) = \{(123)\} \sqcup \{(132)\}$ is the connected component decomposition of $C((123))$. Therefore

$$S_3 = \{(123)\} \sqcup \{(132)\} \sqcup C((12)) \sqcup C(e)$$

is the maximal connected subquandle decomposition of S_3 , and we have $\text{depth}(S_3) = 2$.

(2) *We show that the maximal connected subquandle decomposition of S_4 is*

$$(27) \quad S_4 = C((1234))$$

$$(28) \quad \sqcup \{(123), (142), (134), (243)\} \sqcup \{(132), (124), (143), (234)\} \\ \sqcup \{(12)(34)\} \sqcup \{(13)(24)\} \sqcup \{(14)(23)\} \sqcup C((12)) \sqcup C(e).$$

$S_4 = C((1234)) \sqcup C((123)) \sqcup C((12)(34)) \sqcup C((12)) \sqcup C(e)$ is the connected component decomposition of S_4 . Furthermore, $C((1234))$, $C((12))$ and $C(e)$ are connected quandles, and $C((123)) = \{(123), (142), (134), (243)\} \sqcup \{(132), (124), (143), (234)\}$ and $C((12)(34)) = \{(12)(34)\} \sqcup \{(13)(24)\} \sqcup \{(14)(23)\}$ are the connected component decompositions of $C((123))$ and $C((12)(34))$ respectively. Since any connected component of $C((123))$ and $C((12)(34))$ is connected,

$$(29) \quad S_4 = C((1234))$$

$$(30) \quad \sqcup \{(123), (142), (134), (243)\} \sqcup \{(132), (124), (143), (234)\} \\ \sqcup \{(12)(34)\} \sqcup \{(13)(24)\} \sqcup \{(14)(23)\} \sqcup C((12)) \sqcup C(e)$$

is the maximal connected subquandle decomposition of S_4 , and we have $\text{depth}(S_4) = 2$.

Example 3. *We show that the maximal connected subquandle decomposition of the dihedral quandle R_0 is*

$$R_0 = \bigsqcup_{i \in \mathbb{Z}} \{i\}.$$

We note that R_0 is isomorphic to the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(t+1)$. By Theorem 1, the connected component decomposition of R_0 is

$$R_0 = \text{Orb}(0) \sqcup \text{Orb}(1) = \{i \mid i : \text{even}\} \sqcup \{i \mid i : \text{odd}\}.$$

Since each connected component is isomorphic to R_0 by Corollary 4, we have $D^n(\{R_0\}) = \{\{2^n j + i \mid j \in \mathbb{Z}\} \mid i = 0, 1, \dots, 2^n - 1\}$ for any $n \in \mathbb{Z}_{\geq 0}$ by iterating a connected component decomposition. Hence for any $a, b \in R_0$, there exists $l \in \mathbb{Z}_{>0}$ such that a and b are in distinct elements of $D^l(\{R_0\})$. Since any connected subquandle is included in a connected component, any connected subquandle of R_0 is included in an element of $D^l(\{R_0\})$. Therefore a and b are in distinct maximal connected subquandles of R_0 , which implies that $R_0 = \bigsqcup_{i \in \mathbb{Z}} \{i\}$ is the maximal connected subquandle decomposition of R_0 , and $\text{depth}(R_0) = \infty$.

Example 4. We consider the maximal connected subquandle decomposition of the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)$. Since $\gcd(6, 1^2 + 1 + 1) = 3$, $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1) = \text{Orb}(0) \sqcup \text{Orb}(1) \sqcup \text{Orb}(2)$ is the connected component decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)$, and

$$(31) \quad \text{Orb}(0) = \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], g(1) \equiv 0 \pmod{3}\}$$

$$(32) \quad = \left\{ \begin{array}{l} 0, 3, 3t, 1+2t, 1+5t, 2+t, 2+4t, \\ 3+3t, 4+2t, 4+5t, 5+t, 5+4t \end{array} \right\},$$

$$(33) \quad \text{Orb}(1) = \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], g(1) \equiv 1 \pmod{3}\}$$

$$(34) \quad = \left\{ \begin{array}{l} 1, 4, t, 4t, 1+3t, 2+2t, 2+5t, \\ 3+t, 3+4t, 4+3t, 5+2t, 5+5t \end{array} \right\},$$

$$(35) \quad \text{Orb}(2) = \{[g(t)] \mid g(t) \in \mathbb{Z}[t^{\pm 1}], g(1) \equiv 2 \pmod{3}\}$$

$$= \left\{ \begin{array}{l} 2, 5, 2t, 5t, 1+t, 1+4t, 2+3t, \\ 3+2t, 3+5t, 4+t, 4+4t, 5+3t \end{array} \right\}$$

by Theorem 1

Next, by Theorem 1, for any $i = 0, 1, 2$, $\text{Orb}(i)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((6, t^2 + t + 1) + I)$, where

$$\begin{aligned} I &= \{-a_2(t+2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}], 6a_1 + 3a_2 = 0\} \\ &= \{-a_2(t+2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}], a_2 = -2a_1\} \\ &= 2(t+2)\mathbb{Z}[t^{\pm 1}], \end{aligned}$$

which implies that $\mathbb{Z}[t^{\pm 1}]/((6, t^2 + t + 1) + I) = \mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1)$. By Theorem 1, we obtain that

$$\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1) = \{0, 3, 2+t, 5+t\} \sqcup \{1, 4, t, 3+t\} \sqcup \{2, 5, 1+t, 4+t\}$$

is the connected component decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1)$. By the proof of Theorem 1, the map $\tilde{\phi} : \mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2 + t + 1) \rightarrow (1-t)(\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1))$ defined by $\tilde{\phi}(x) = (1-t)x$ is an isomorphism. Hence, by the proof of Lemma 1,

$$\begin{aligned} \text{Orb}(0) &= (1-t)(\mathbb{Z}[t^{\pm 1}]/(6, t^2 + t + 1)) \\ &= \{0, 3, 3t, 3+3t\} \sqcup \{1+5t, 1+2t, 4+2t, 4+5t\} \sqcup \{2+4t, 2+t, 5+t, 5+4t\} \end{aligned}$$

is the connected component decomposition of $\text{Orb}(0)$. Furthermore, for any $i = 1, 2$, the map $\phi_i : \text{Orb}(0) \rightarrow \text{Orb}(i)$ defined by $\phi_i(x) = x + i$ is an isomorphism by the proof of Lemma 1. Therefore

$$\text{Orb}(1) = \{1, 4, 1 + 3t, 4 + 3t\} \sqcup \{2 + 5t, 2 + 2t, 5 + 2t, 5 + 5t\} \sqcup \{3 + 4t, 3 + t, t, 4t\}$$

and

$$\text{Orb}(2) = \{2, 5, 2 + 3t, 5 + 3t\} \sqcup \{3 + 5t, 3 + 2t, 2t, 5t\} \sqcup \{4 + 4t, 4 + t, 1 + t, 1 + 4t\}$$

are the connected component decompositions of $\text{Orb}(1)$ and $\text{Orb}(2)$ respectively.

Finally, by Theorem 1, any connected component of $\mathbb{Z}[t^{\pm 1}]/(6, 2(t+2), t^2+t+1)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/((6, 2(t+2), t^2+t+1) + I')$, where

$$\begin{aligned} I' &= \{-2a_2 - a_3(t+2) \mid a_1, a_2, a_3 \in \mathbb{Z}[t^{\pm 1}], 6a_1 + 6a_2 + 3a_3 = 0\} \\ &= \{-2a_2 - a_3(t+2) \mid a_1, a_2, a_3 \in \mathbb{Z}[t^{\pm 1}], a_3 = -2(a_1 + a_2)\} \\ &= \{-2a_2 + 2(a_1 + a_2)(t+2) \mid a_1, a_2 \in \mathbb{Z}[t^{\pm 1}]\} \\ &= 2\mathbb{Z}[t^{\pm 1}], \end{aligned}$$

which implies that $\mathbb{Z}[t^{\pm 1}]/((6, 2(t+2), t^2+t+1) + I') = \mathbb{Z}[t^{\pm 1}]/(2, t^2+t+1)$. By Corollary 2, $\mathbb{Z}[t^{\pm 1}]/(2, t^2+t+1)$ is a connected quandle. Therefore

$$(3\mathbb{Z}[t^{\pm 1}]/(6, t^2+t+1))$$

$$(38) \{0, 3, 3t, 3 + 3t\} \sqcup \{1 + 5t, 1 + 2t, 4 + 2t, 4 + 5t\} \sqcup \{2 + 4t, 2 + t, 5 + t, 5 + 4t\}$$

$$(39) \sqcup \{1, 4, 1 + 3t, 4 + 3t\} \sqcup \{2 + 5t, 2 + 2t, 5 + 2t, 5 + 5t\} \sqcup \{3 + 4t, 3 + t, t, 4t\}$$

$$\sqcup \{2, 5, 2 + 3t, 5 + 3t\} \sqcup \{3 + 5t, 3 + 2t, 2t, 5t\} \sqcup \{4 + 4t, 4 + t, 1 + t, 1 + 4t\}$$

is the maximal connected subquandle decomposition of $\mathbb{Z}[t^{\pm 1}]/(6, t^2+t+1)$, and we obtain that $\text{depth}(\mathbb{Z}[t^{\pm 1}]/(6, t^2+t+1)) = 2$.

Proposition 1. Let $n_0 \in \mathbb{Z}_{>0}$, $a \in \mathbb{Z}$ and put $n_{i+1} := n_i / \gcd(n_i, 1+a)$ for any $i \in \mathbb{Z}_{\geq 0}$. Let l be the minimal number satisfying $n_l = n_{l+1}$. Then the Alexander quandle $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into N maximal connected subquandles, where $N = \prod_{i=0}^{l-1} \gcd(n_i, 1+a)$, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_l, t+a)$.

Proof. By Theorem 1, for any $i \in \mathbb{Z}_{\geq 0}$, $\mathbb{Z}[t^{\pm 1}]/(n_i, t+a)$ is decomposed into $\gcd(n_i, 1+a)$ maximal connected subquandles, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_i, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_{i+1}, t+a)$. Hence for any $i \in \mathbb{Z}_{>0}$, any element of $D^i(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\})$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_i, t+a)$, and we have $\#D^i(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\}) = \prod_{j=0}^{i-1} \gcd(n_j, 1+a)$. By Corollary 2, $n_k = n_{k+1}$, that is, $\gcd(n_k, 1+a) = 1$ if and only if $D^k(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\}) = D^{k+1}(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\})$. Hence l is the minimal number satisfying $n_l = n_{l+1}$ if and only if $\text{depth}(\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)) = l$. By Theorem 3, $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a) = \bigsqcup_{C \in D^l(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\})} C$ is the maximal connected subquandle decomposition of $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$. Since $N = \prod_{i=0}^{l-1} \gcd(n_i, 1+a) = \#D^l(\{\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)\})$, $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into N maximal connected subquandles, and any maximal connected subquandle of $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n_l, t+a)$. \square

In Proposition 1, if $1+a$ is a prime number, $\mathbb{Z}[t^{\pm 1}]/(n_0, t+a)$ is decomposed into $|1+a|^l$ maximal connected subquandles, and any maximal connected subquandle

of $\mathbb{Z}[t^{\pm 1}]/(n_0, t + a)$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(k, t + a)$, where $n_0 = k(1 + a)^l$ such that k and $1 + a$ are relatively prime integers.

By Proposition 1, the following corollary holds.

Corollary 6. *For any $m \in \mathbb{Z}_{>0}$, the dihedral quandle R_m is decomposed into 2^l maximal connected subquandles, and any maximal connected subquandle of R_m is isomorphic to R_k , and $\text{depth}(R_m) = l$, where k is an odd number, and $l \in \mathbb{Z}_{>0}$ such that $m = 2^l k$.*

6. A MULTIPLE CONJUGATION QUANDLE AND A G -FAMILY OF QUANGLES

We recall the definition of a multiple conjugation quandle [4].

Definition 1. *A multiple conjugation quandle X is a disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $*$: $X \times X \rightarrow X$ satisfying the following axioms.*

- *For any $a, b \in G_\lambda$, $a * b = b^{-1}ab$.*
- *For any $x \in X$ and $a, b \in G_\lambda$, $x * e_\lambda = x$ and $x * (ab) = (x * a) * b$, where e_λ is the identity of G_λ .*
- *For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.*
- *For any $x \in X$ and $a, b \in G_\lambda$, $(ab) * x = (a * x)(b * x)$, where $a * x, b * x \in G_\mu$ for some $\mu \in \Lambda$.*

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle. In this paper, we write $a_1 * a_2 * \cdots * a_n$ for $(\cdots ((a_1 * a_2) * a_3) * \cdots * a_n)$ simply and denote by G_a the group G_λ containing $a \in X$. we also denote by e_λ the identity of G_λ . Then the identity of G_a is denoted by e_a for any $a \in X$. For any $a, b \in X$, we define a map $S_a : X \rightarrow X$ by $S_a(x) = x * a$ and a binary operation $*^{-1} : X \times X \rightarrow X$ by $a *^{-1} b = S_b^{-1}(a)$.

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda, Y = \bigsqcup_{\mu \in M} G_\mu$ and $(X, *_X), (Y, *_Y)$ be multiple conjugation quandles. A *homomorphism* $\phi : X \rightarrow Y$ is a map from X to Y satisfying $\phi(x *_X y) = \phi(x) *_Y \phi(y)$ for any $x, y \in X$ and $\phi(ab) = \phi(a)\phi(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. We call a bijective homomorphism an *isomorphism*. X and Y are *isomorphic*, denoted by $X \cong Y$, if there exists an isomorphism from X to Y . We call an isomorphism from X to X an *automorphism* of X . For any $a \in X$, the map S_a is an automorphism of X .

Let $(X, *)$ be a multiple conjugation quandle. A non-empty subset Y of X is called a *sub-multiple conjugation quandle* of X if Y itself is a multiple conjugation quandle under $*$ and the group operations of X .

Proposition 2. *Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle and let Y be a non-empty subset of X . Then the following are equivalent.*

- (1) *Y is a sub-multiple conjugation quandle of X .*
- (2) *For any $a, b \in Y$, $a * b \in Y$, and $Y \cap G_\lambda$ is a subgroup of G_λ or empty set for any $\lambda \in \Lambda$.*
- (3) *For any $a, b \in Y$, $a * b \in Y$, and there exists a subset $\Lambda' \subset \Lambda$ such that $Y = \bigsqcup_{\lambda \in \Lambda'} H_\lambda$, where H_λ is a subgroup of G_λ for any $\lambda \in \Lambda'$.*

Proof. First suppose that it satisfies (1). Then $a * b \in Y$ for any $a, b \in Y$ and $Y = \bigsqcup_{\mu \in \Lambda'} H_\mu$, where H_μ is a group for any $\mu \in \Lambda'$. For any $\mu \in \Lambda'$, there uniquely exists $\lambda \in \Lambda$ such that H_μ is a subgroup of G_λ . Then we again write H_λ for H_μ . Since $H_\lambda \cap H_{\lambda'} = \emptyset$ ($\lambda \neq \lambda'$) for any $\lambda, \lambda' \in \Lambda'$, we have $Y \cap G_\lambda = H_\lambda$. Hence H_λ is a subgroup of G_λ for any $\lambda \in \Lambda'$, and we have $\Lambda' \subset \Lambda$. Therefore it satisfies (3).

Second suppose that it satisfies (3). Since $G_\lambda \cap G_{\lambda'} = \emptyset$ and $H_\mu \cap H_{\mu'} = \emptyset$ when $\lambda \neq \lambda'$ and $\mu \neq \mu'$ for any $\lambda, \lambda' \in \Lambda$ and $\mu, \mu' \in \Lambda'$, we have $Y \cap G_\mu = H_\mu$ for any $\mu \in \Lambda'$ and $Y \cap G_\lambda = \emptyset$ for any $\lambda \in \Lambda - \Lambda'$. Therefore it satisfies (2). Finally suppose that it satisfies (2). Let $\Lambda' := \{\lambda \in \Lambda \mid Y \cap G_\lambda \neq \emptyset\}$. Since $G_\lambda \cap G_{\lambda'} = \emptyset$ when $\lambda \neq \lambda'$ for any $\lambda, \lambda' \in \Lambda$, we have $Y = \bigsqcup_{\lambda \in \Lambda'} (Y \cap G_\lambda)$, where $Y \cap G_\lambda$ is a subgroup of G_λ for any $\lambda \in \Lambda'$. Y clearly satisfies the axioms of a multiple conjugation quandle. Therefore it satisfies (1). \square

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle and let A be a subset of X . Then the minimal sub-multiple conjugation quandle of X including A , denote by $\langle A \rangle_{\text{MCQ}}$, is called the sub-multiple conjugation quandle generated by A .

We recall the definition of a G -family of quandles [3].

Definition 2. Let G be a group with the identity element e . A G -family of quandles is a non-empty set X with a family of binary operations $*^g : X \times X \rightarrow X (g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x *^{gh} y = (x *^g y) *^h y$ and $x *^e y = x$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

Then the following proposition holds.

Proposition 3 ([3]). Let $(X, *)$ be a quandle and let m be the type of X . Then $(X, \{*\^i\}_{i \in \mathbb{Z}_{km}})$ (resp. $(X, \{*\^i\}_{i \in \mathbb{Z}})$) is a \mathbb{Z}_{km} (resp. \mathbb{Z})-family of quandles for any $k \in \mathbb{Z}_{>0}$.

Let $(X, \{*\^g\}_{g \in G})$ be a G -family of quandles. Then $\bigsqcup_{x \in X} \{x\} \times G$ is a multiple conjugation quandle with

$$(x, g) * (y, h) := (x *^h y, h^{-1}gh), \quad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$. We call it the *associated multiple conjugation quandle* of $(X, \{*\^g\}_{g \in G})$.

7. THE MAXIMAL CONNECTED SUB-MULTIPLE CONJUGATION QUANDLE DECOMPOSITION

In this section, we consider a decomposition of a multiple conjugation quandle into the disjoint union of maximal connected sub-multiple conjugation quandles, and show that it is uniquely obtained by iterating a connected component decomposition when the multiple conjugation quandle is the finite disjoint union of groups.

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ which is a multiple conjugation quandle. All automorphisms of X form a group under composition of morphisms: $f \cdot g := g \circ f$. This group is called the *automorphism group* of X and denoted by $\text{Aut}(X)$. For a subset A of X , we denote by $\text{Inn}(A)$ a subgroup of $\text{Aut}(X)$ generated by $\{S_a \mid a \in A\}$. In particular, $\text{Inn}(X)$ is called the *inner automorphism group* of X . For any $\lambda \in \Lambda$ and $g \in \text{Inn}(A)$, we define an action of $\text{Inn}(A)$ on Λ by $e_{\lambda \cdot g} = g(e_\lambda)$. We say that X is a *connected multiple conjugation quandle* when the action is transitive. In this paper, we call an orbit of Λ by the action an orbit of Λ simply and $\Lambda = \bigsqcup_{i \in I} \Lambda_i$ the orbit decomposition of Λ when Λ_i is an orbit of Λ for any $i \in I$. We denote by $\text{Orb}_\Lambda(\lambda)$ or $\text{Orb}(\lambda)$ the orbit of Λ containing λ . In general, for any orbit Λ' of Λ , $\bigsqcup_{\lambda \in \Lambda'} G_\lambda$ is a sub-multiple conjugation quandle of X and called a *connected*

component of X . However $\bigsqcup_{\lambda \in \Lambda'} G_\lambda$ is not a connected sub-multiple conjugation quandle of X in general.

Proposition 4. *Let X be a quandle, Y be a subquandle of X , m be the type of X and let $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$ be the associated multiple conjugation quandle of a \mathbb{Z}_m -family of quandles $(X, \{*\}^i_{i \in \mathbb{Z}_m})$. Then Y is a connected component of X if and only if $\bigsqcup_{x \in Y} \{x\} \times \mathbb{Z}_m$ is a connected component of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$.*

Proof. Suppose that for any $a, b \in Y$, there exists $f \in \text{Inn}(X)$ such that $f(a) = b$. Then for any $(a, 0), (b, 0) \in \bigsqcup_{x \in Y} \{x\} \times \mathbb{Z}_m$, there exists $f = S_{x_n}^{k_n} \circ \dots \circ S_{x_2}^{k_2} \circ S_{x_1}^{k_1} \in \text{Inn}(X)$ such that $f(a) = b$. Hence there exists $g = S_{(x_n, k_n)} \circ \dots \circ S_{(x_2, k_2)} \circ S_{(x_1, k_1)} \in \text{Inn}(\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m)$ such that $g((a, 0)) = (b, 0)$. On the other hand, suppose that for any $a, b \in Y$, there exists $g \in \text{Inn}(\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m)$ such that $g((a, 0)) = (b, 0)$. Then for any $a, b \in Y$, there exists $g = S_{(x_n, k_n)}^{i_n} \circ \dots \circ S_{(x_2, k_2)}^{i_2} \circ S_{(x_1, k_1)}^{i_1} \in \text{Inn}(\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m)$ such that $g((a, 0)) = (b, 0)$. Hence there exists $f = S_{x_n}^{i_n k_n} \circ \dots \circ S_{x_2}^{i_2 k_2} \circ S_{x_1}^{i_1 k_1} \in \text{Inn}(X)$ such that $f(a) = b$. Therefore Y is a connected component of X if and only if $\bigsqcup_{x \in Y} \{x\} \times \mathbb{Z}_m$ is a connected component of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$. \square

By Proposition 4, we obtain the following corollary immediately.

Corollary 7. *Let X be a quandle, m be the type of X and let $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$ be the associated multiple conjugation quandle of a \mathbb{Z}_m -family of quandles $(X, \{*\}^i_{i \in \mathbb{Z}_m})$. Then X is a connected quandle if and only if $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$ is a connected multiple conjugation quandle.*

Let X be a multiple conjugation quandle and let A be a connected sub-multiple conjugation quandle of X . We say that A is a *maximal connected sub-multiple conjugation quandle* of X when any connected sub-multiple conjugation quandle of X including A is only A . We say that $X = \bigsqcup_{i \in I} A_i$ is the *maximal connected sub-multiple conjugation quandle decomposition* of X when each A_i is a maximal connected sub-multiple conjugation quandle of X . By Corollary 7, we obtain the following corollary.

Corollary 8. *Let X be a quandle, m be the type of X and let $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$ be the associated multiple conjugation quandle of a \mathbb{Z}_m -family of quandles $(X, \{*\}^i_{i \in \mathbb{Z}_m})$. Then the following hold.*

- (1) *A is a maximal connected subquandle of X if and only if $\bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$ is a maximal connected sub-multiple conjugation quandle of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$.*
- (2) *$X = \bigsqcup_{i \in I} A_i$ is the maximal connected subquandle decomposition of X if and only if $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m = \bigsqcup_{i \in I} (\bigsqcup_{x \in A_i} \{x\} \times \mathbb{Z}_m)$ is the maximal connected sub-multiple conjugation quandle decomposition of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$.*

Proof. It is sufficient to prove (1) since (2) follows from (1) immediately. Suppose that A is a maximal connected subquandle of X . Let Y be a connected sub-multiple conjugation quandle including $\bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$. Then there exists a connected subquandle B of X including A such that $Y = \bigsqcup_{x \in B} \{x\} \times \mathbb{Z}_m$ by Corollary 7. Hence we have $A = B$ and $Y = \bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$. Therefore $\bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$ is a maximal connected sub-multiple conjugation quandle of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$. On the other

hand, suppose that $\bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$ is a maximal connected sub-multiple conjugation quandle of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$. Let B be a connected subquandle including A and let $Y = \bigsqcup_{x \in B} \{x\} \times \mathbb{Z}_m$. By Corollary 7, Y is a connected sub-multiple conjugation quandle of $\bigsqcup_{x \in X} \{x\} \times \mathbb{Z}_m$. Hence we have $Y = \bigsqcup_{x \in A} \{x\} \times \mathbb{Z}_m$ and $A = B$. Therefore A is a maximal connected subquandle of X . \square

Example 5. Let $m \in \mathbb{Z}_{>0}$. Since the dihedral quandle R_m is of type 2, $\bigsqcup_{x \in R_m} \{x\} \times \mathbb{Z}_2$ is the associated multiple conjugation quandle of the \mathbb{Z}_2 -family of quandles $(R_m, \{*\}^i)_{i \in \mathbb{Z}_2}$. By Corollary 6, R_m is decomposed into 2^l maximal connected subquandles, and any maximal connected subquandle of R_m is isomorphic to R_k , where k is an odd number, and $l \in \mathbb{Z}_{>0}$ such that $m = 2^l k$. Therefore $\bigsqcup_{x \in R_m} \{x\} \times \mathbb{Z}_2$ is decomposed into 2^l maximal connected sub-multiple conjugation quandles, and any maximal connected sub-multiple conjugation quandles of $\bigsqcup_{x \in R_m} \{x\} \times \mathbb{Z}_2$ is isomorphic to $\bigsqcup_{x \in R_k} \{x\} \times \mathbb{Z}_2$ by Corollary 8.

Let X be a multiple conjugation quandle and let A be a subset of X . For any $a, b \in X$, we write $a \stackrel{\text{MCQ}}{\sim}_A b$ when there exists $f \in \text{Inn}(A)$ such that $f(e_a) = e_b$. It is an equivalence relation on X . It is easy to see that $a \stackrel{\text{MCQ}}{\sim}_A b$ if and only if there exist $a_1, a_2, \dots, a_n \in A$ and $k_1, k_2, \dots, k_n \in \mathbb{Z}$ such that $e_a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n = e_b$. Furthermore X is a connected multiple conjugation quandle if and only if $a \stackrel{\text{MCQ}}{\sim}_X b$ for any $a, b \in X$.

Lemma 7. Let X be a multiple conjugation quandle and let A be a subset of X . Then for any $x \in \langle A \rangle_{\text{MCQ}}$, there exists $a \in A$ such that $x \stackrel{\text{MCQ}}{\sim}_A a$.

Proof. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$, $\Lambda' = \{\lambda \in \Lambda \mid \langle A \rangle \cap G_\lambda \neq \emptyset\}$, where $\langle A \rangle$ is a subquandle of X generated by A , and let H_λ be a subgroup of G_λ generated by $\langle A \rangle \cap G_\lambda$ for any $\lambda \in \Lambda'$. For any $x, y \in \bigsqcup_{\lambda \in \Lambda'} H_\lambda$, there exist $\lambda_1, \lambda_2 \in \Lambda'$ such that $x \in H_{\lambda_1}, y \in H_{\lambda_2}$, and there exist $w_1, w_2, \dots, w_n \in \langle A \rangle \cap G_{\lambda_1}, v_1, v_2, \dots, v_m \in \langle A \rangle \cap G_{\lambda_2}$ and $k_1, k_2, \dots, k_n, l_1, l_2, \dots, l_m \in \mathbb{Z}$ such that $x = w_1^{k_1} w_2^{k_2} \dots w_n^{k_n}$ and $y = v_1^{l_1} v_2^{l_2} \dots v_m^{l_m}$. Then we have $x * y = (w_1^{k_1} w_2^{k_2} \dots w_n^{k_n}) * (v_1^{l_1} v_2^{l_2} \dots v_m^{l_m}) = (w_1 *^{l_1} v_1 *^{l_2} \dots *^{l_m} v_m)^{k_1} (w_2 *^{l_1} v_1 *^{l_2} \dots *^{l_m} v_m)^{k_2} \dots (w_n *^{l_1} v_1 *^{l_2} \dots *^{l_m} v_m)^{k_n}$. Since $\langle A \rangle$ is a quandle, $w_i *^{l_1} v_1 *^{l_2} \dots *^{l_m} v_m \in \langle A \rangle$ for any $i = 1, 2, \dots, n$, and then there exists $\lambda_0 \in \Lambda$ such that $w_i *^{l_1} v_1 *^{l_2} \dots *^{l_m} v_m \in G_{\lambda_0}$ for any $i = 1, 2, \dots, n$. Since $\langle A \rangle \cap G_{\lambda_0} \neq \emptyset$, we have $\lambda_0 \in \Lambda'$ and $x * y \in H_{\lambda_0} \subset \bigsqcup_{\lambda \in \Lambda'} H_\lambda$. Hence $\bigsqcup_{\lambda \in \Lambda'} H_\lambda$ is a sub-multiple conjugation quandle of X by Proposition 2. Then we have $\langle A \rangle_{\text{MCQ}} \subset \bigsqcup_{\lambda \in \Lambda'} H_\lambda$ since $A \subset \bigsqcup_{\lambda \in \Lambda'} H_\lambda$. Hence for any $x \in \langle A \rangle_{\text{MCQ}}$, there exists $\mu \in \Lambda'$ such that $x \in H_\mu$ and $\langle A \rangle \cap G_\mu \neq \emptyset$. Consequently, there exist $a, a_1, a_2, \dots, a_n \in A$ and $k_1, k_2, \dots, k_n \in \mathbb{Z}$ such that $a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n \in \langle A \rangle \cap G_\mu$. Therefore we have $e_x = (a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n)(a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n)^{-1} = (aa^{-1}) *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n = e_a *^{k_1} a_1 *^{k_2} \dots *^{k_n} a_n$, which implies that $x \stackrel{\text{MCQ}}{\sim}_A a$. \square

Lemma 8. Let X be a multiple conjugation quandle and let A_i be connected sub-multiple conjugation quandles of X for any $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\langle \bigcup_{i \in I} A_i \rangle_{\text{MCQ}}$ is a connected sub-multiple conjugation quandle of X .

Proof. Let $A = \bigcup_{i \in I} A_i$, and suppose that $\bigcap_{i \in I} A_i \neq \emptyset$. For any $x, y \in \langle A \rangle_{\text{MCQ}}$, there exist $a, b \in A$ such that $x \stackrel{\text{MCQ}}{\sim}_A a$ and $y \stackrel{\text{MCQ}}{\sim}_A b$ by Lemma 7. For any $c \in \bigcap_{i \in I} A_i$, we have $a \stackrel{\text{MCQ}}{\sim}_A c$ and $b \stackrel{\text{MCQ}}{\sim}_A c$. Hence we have $a \stackrel{\text{MCQ}}{\sim}_A b$, which

implies that $x \stackrel{\text{MCQ}}{\sim}_A y$. Therefore $x \stackrel{\text{MCQ}}{\sim}_{\langle A \rangle_{\text{MCQ}}} y$, that is, $\langle A \rangle_{\text{MCQ}}$ is a connected sub-multiple conjugation quandle of X . \square

Lemma 9. *Let X be a multiple conjugation quandle and let A_1 and A_2 be maximal connected sub-multiple conjugation quandles of X . If $A_1 \cap A_2 \neq \emptyset$, then $A_1 = A_2$.*

Proof. Suppose that $A_1 \cap A_2 \neq \emptyset$. By Lemma 8, $\langle A_1 \cup A_2 \rangle_{\text{MCQ}}$ is a connected sub-multiple conjugation quandle of X . Since A_1 and A_2 are included in $\langle A_1 \cup A_2 \rangle_{\text{MCQ}}$ and maximal connected sub-multiple conjugation quandles of X , we obtain $\langle A_1 \cup A_2 \rangle_{\text{MCQ}} = A_1 = A_2$. \square

Theorem 4. *Any multiple conjugation quandle has the unique maximal connected sub-multiple conjugation quandle decomposition.*

Proof. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle. For any $a \in X$, we define $[a] := \bigcup_{a \in W} W$, where W is a connected sub-multiple conjugation quandle of X . By Lemma 8, $\langle [a] \rangle_{\text{MCQ}}$ is a connected sub-multiple conjugation quandle of X . Suppose that A is a connected sub-multiple conjugation quandle of X including $\langle [a] \rangle_{\text{MCQ}}$. Since A contains a , we have $A \subset [a]$, which implies that $A = [a] = \langle [a] \rangle_{\text{MCQ}}$. Hence $[a]$ is a maximal connected sub-multiple conjugation quandle of X . For any $a, b \in X$, we have $[a] \cap [b] = \emptyset$ or $[a] = [b]$ by Lemma 9. Therefore there exists a subset Y of X such that $X = \bigsqcup_{a \in Y} [a]$. It is the maximal connected sub-multiple conjugation quandle decomposition of X . Next, we show the uniqueness. Let B be a maximal connected sub-multiple conjugation quandle of $X = \bigsqcup_{a \in Y} [a]$. Then there exists $a' \in Y$ such that $B \cap [a'] \neq \emptyset$. By Lemma 9, we have $B = [a']$. Therefore X has the unique maximal connected sub-multiple conjugation quandle decomposition. \square

For a multiple conjugation quandle X , it is easy to see that any connected sub-multiple conjugation quandle of X is included in some connected component of X . Therefore if a connected component of X is a connected sub-multiple conjugation quandle of X , then it is a maximal connected sub-multiple conjugation quandle of X .

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle and let $\mathcal{P}(\Lambda)$ be the set of all subsets of Λ . For any $\mathcal{A} \subset \mathcal{P}(\Lambda)$, we define $D(\mathcal{A}) := \bigcup_{\Lambda \in \mathcal{A}} \{\text{Orb}_\Lambda(\mu) \mid \mu \in \Lambda\}$. It is easy to see that $\bigcup_{\Lambda \in \mathcal{A}} \Lambda = \bigcup_{\Lambda \in D(\mathcal{A})} \Lambda$. We put $D^0(\mathcal{A}) := \mathcal{A}$ and $D^{k+1}(\mathcal{A}) := D(D^k(\mathcal{A}))$ for any $k \in \mathbb{Z}_{\geq 0}$.

Theorem 5. *Let X be a multiple conjugation quandle. If there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(\{\Lambda\}) = D^{n+1}(\{\Lambda\})$, then $X = \bigsqcup_{\Lambda' \in D^n(\{\Lambda\})} (\bigsqcup_{\lambda \in \Lambda'} G_\lambda)$ is the maximal connected sub-multiple conjugation quandle decomposition of X . In particular, if Λ is a finite set, then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $X = \bigsqcup_{\Lambda' \in D^n(\{\Lambda\})} (\bigsqcup_{\lambda \in \Lambda'} G_\lambda)$ is the maximal connected sub-multiple conjugation quandle decomposition of X .*

Proof. Suppose that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(\{\Lambda\}) = D^{n+1}(\{\Lambda\})$. For any $\Lambda' \in D^n(\{\Lambda\})$, Λ' has only one orbit Λ' , that is, $\bigsqcup_{\lambda \in \Lambda'} G_\lambda$ is a connected sub-multiple conjugation quandle of X . Let $Y = \bigsqcup_{\lambda \in M} H_\lambda$ be a connected sub-multiple conjugation quandle of X including $\bigsqcup_{\lambda \in \Lambda'} G_\lambda$, where H_λ is a subgroup of G_λ for any $\lambda \in M$, and let $\Lambda_0 = \Lambda$. Then there exists an orbit of Λ_0 by the action of $\text{Inn}(\bigsqcup_{\lambda \in \Lambda_0} G_\lambda)$ including M . We denote by $\Lambda_1 \in D^1(\{\Lambda\})$ the orbit. For any $i \in \mathbb{Z}_{\geq 0}$, we denote by $\Lambda_{i+1} \in D^{i+1}(\{\Lambda\})$ an orbit of Λ_i by the action of $\text{Inn}(\bigsqcup_{\lambda \in \Lambda_i} G_\lambda)$

including M inductively. Then we have $\Lambda' \subset M \subset \Lambda_n$. Since $\Lambda', \Lambda_n \in D^n(\{\Lambda\})$, we have $\Lambda' = M = \Lambda_n$ and $Y = \bigsqcup_{\lambda \in \Lambda'} H_\lambda$. Since H_λ is a subgroup of G_λ for any $\lambda \in \Lambda'$, we have $Y = \bigsqcup_{\lambda \in \Lambda'} G_\lambda$. Therefore $\bigsqcup_{\lambda \in \Lambda'} G_\lambda$ is a maximal connected sub-multiple conjugation quandle of X , and $X = \bigsqcup_{\Lambda' \in D^n(\{\Lambda\})} (\bigsqcup_{\lambda \in \Lambda'} G_\lambda)$ is the maximal connected sub-multiple conjugation quandle decomposition of X . Next, let Λ be a finite set. For any $i \in \mathbb{Z}_{\geq 0}$, we have $\#D^i(\{\Lambda\}) \leq \#D^{i+1}(\{\Lambda\}) \leq \#\Lambda$. Since $\#\Lambda$ is finite, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\#D^n(\{\Lambda\}) = \#D^{n+1}(\{\Lambda\})$, which implies that $D^n(\{\Lambda\}) = D^{n+1}(\{\Lambda\})$. Therefore $X = \bigsqcup_{\Lambda' \in D^n(\{\Lambda\})} (\bigsqcup_{\lambda \in \Lambda'} G_\lambda)$ is the maximal connected sub-multiple conjugation quandle decomposition of X . \square

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